# ON A CLASS OF MINIMAX PROBLEMS WITH DIFFERENTIAL CONSTRAINTS

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**1.** We consider a system of differential equations

$$dz^i / dx = f^i(z, x, a) \quad (i = 1, \ldots, n) \tag{1.1}$$

containing in its right-hand side parameters $a(a_1, \ldots, a_r)$. Initial conditions

$$z^i(0) = g^i(a) \quad (i = 1, \ldots, n) \tag{1.2}$$

may also depend on parameters $a$ which should be chosen so, as to minimize the functional

$$I = \max_x |F(z^1, \ldots, z^n; x; a_1, \ldots, a_r)|, \quad x \in [0, l] \tag{1.3}$$

We assume that $f^i$ and $F$ possess continuous derivatives in $z$ and $a$ up to the second order.

Solution of this problem which follows, is preceded by an auxilliary construction unrelated to the functional (1.3). Namely, using the notation

$$p_k^i = \partial z^i / \partial a_k \qquad (k = 1, \ldots, r) \tag{1.4}$$

we construct Eqs. (see (1.1) and (1.2))

$$\frac{dp_k^i}{dx} = f^i_{a_k} + \sum_{j=1}^{n} f^i_{z^j} p_k^j \tag{1.5}$$

with the corresponding initial conditions

$$p_k^i(0) = \partial g^i / \partial a_k \tag{1.6}$$

Introducing now a coupled system (a system for multipliers)

$$\frac{d\lambda_i}{dx} = -\sum_{j=1}^{n} \lambda_j f^j_{z^i} \qquad (i = 1, \ldots, n) \tag{1.7}$$

we shall multiply (1.5) by $\lambda_i$ and (1.7) by $p_k^i$. Summing over $i$ and collecting like terms, we obtain

$$\frac{d}{dx} \sum_{i=1}^{n} \lambda_i p_k^i = \sum_{i=1}^{n} \lambda_i f^i_{a_k} \qquad (k = 1, \ldots, r) \tag{1.8}$$

which plays a major role in solving various problems of optimum control. To minimize, for example, the value of $z^s(l)$ (Meyer's problem), we integrate (1.8) from 0 to $l$, to obtain

$$\Big(\sum_{i=1}^{n} \lambda_i p_k^i\Big)_{x=l} - \Big(\sum_{i=1}^{n} \lambda_i p_k^i\Big)_{x=0} = \int_0^l \sum_{i=1}^{n} \lambda_i f^i_{a_k}\, dx$$

Putting

$$\lambda_i|_{x=l} = \delta_i^s \qquad (\delta_i^s \text{ is a Kronecker delta})$$

we find

$$p_k^s|_{x=l} = \Big(\sum_{i=1}^{n} \lambda_i p_k^i\Big)_{x=0} + \int_0^l \sum_{i=1}^{n} \lambda_i f^i_{a_k}\, dx$$

Minimum value of $z^s(\ell)$ is stationary in $a_k$ ($k=1,\ldots,r$), hence

$$\left(\sum_{i=1}^{n} \lambda_i p_k{}^i\right)_{x=0} + \int_0^l \sum_{i=1}^{n} \lambda_i f_{a_k}{}^i \, dx = 0 \qquad (k=1,\ldots,r) \tag{1.9}$$

follow, which constitute the necessary conditions for a minimum. Necessary conditions of second order can also be obtained without difficulty.

Returning to our problem we assume, that the absolute maximum of the function $|F|$ relative to $x$, can be minimized by the parameter $a^0$ ($a_1{}^0,\ldots, a_r{}^0$).

Let us assume that for this value of the parameter, the required maximum is reached at the points $x^{(1)}, x^{(2)}, \ldots,$ the set of which may be finite or infinite.

In order to calculate the value of $p_k^s$ at th point $x^{(\nu)}$, we shall introduce initial conditions

$$\lambda_i|_{x=x^{(\nu)}} = \delta_i{}^s \tag{1.10}$$

and we shall denote, under these conditions, the integrals of Eqs. (1. 7), by $\lambda_i{}^{s(\nu)}$. Formula (1. 8) now yields

$$p_k{}^s|_{x=x^{(\nu)}} = \left(\sum_{i=1}^{n} \lambda_i^{s(\nu)} p_k{}^i\right)_{x=0} + \int_0^{x^{(\nu)}} \sum_{i=1}^{n} \lambda_i^{s(\nu)} f_{a_k}{}^i \, dx \tag{1.11}$$

At this stage we must apply a general criterion given in 1943 by Chebotarev [1] for the minimax problem of a given function of two sets of variables. Namely, let the function

$$\varphi\,(x_1, \ldots, x_n;\, a_1, \ldots, a_r) \tag{1.12}$$

of arguments $x(x_1,\ldots,x_n)$ and parameters $a(a_1,\ldots,a_r)$ be: (a) bounded and possessing continuous partial derivatives of first two orders with respect to parameters and (b) let the points $x$ satisfying the inequality $\varphi(x,a) > \varphi_0$ where $\varphi_0$ is a constant and $a$ is in some vicinity of $a^0$, form a compact set.

Let the absolute maximum $\Phi(a)$ of (1. 12) with respect to the arguments be attained at the points $x^{(1)}, x^{(2)}, \ldots$ and let the value $a = a^0$ of the parameter minimize this maximum. Further, denote by $Y^{(\nu)}$ an $r$-dimensional vector whose components are

$$y_k{}^{(\nu)} = \frac{\partial \varphi\,(x, a^0)}{\partial a_k}\bigg|_{x=x^{(\nu)}} \qquad (k=1,\ldots,r) \tag{1.13}$$

Then the following theorems giving, respectively, the sufficient and necessary conditions of minimax, are true.

T h e o r e m 1. If the function (1. 12) satisfies condition (a) and if, for any vector $C$ with components $c_1,\ldots, c_r$ such a pair of vectors $Y^{(\mu)}$ and $Y^{(\nu)}$, can be found that the scalar products $CY^{(\mu)}$ and $CY^{(\nu)}$ are of opposite sign, then the function $\Phi(a)$ has a minimum at the point $a = a^0$.

T h e o r e m 2. If the function (1. 12) satisfies conditions (a) and (b) and if such vector $C$ exists that all scalar products $CY^{(\nu)}$ ($\nu = 1, 2, \ldots$) are of the same sign, then the function $\Phi(a)$ has no minimum at $a = a^0$.

Assertion of both theorems are unified in the requirement [1 and 2] that the system of linear Eqs.

$$\sum_{\nu} m_\nu y_k{}^{(\nu)} = 0 \qquad (k=1,\ldots,r) \tag{1.14}$$

has positive solutions in $m_\nu$.

There exists a case not covered by the above theorems, when we have a vector $C$ for which $CY^{(\nu)} \geqslant 0$ for all $\nu$, but not a vector $C$ for which $CY^{(\nu)} > 0$. Then, the equivalent formulation is as follows: let (1. 14) have nonnegative solutions and out of them, let $m_1, m_2, \ldots, m_p$ allow positive solutions, and the remaining $m_\nu$ null solutions. If rank of the matrix

$$\| y_i^{(\nu)} \| \; (i = 1, \ldots, r;\ (\nu) = (1), \ldots, (p)) \tag{1.15}$$

where $i$ is the row index equal to $r$, then the function $\varphi(x, a)$ has a minimax at $a = a^0$.

To apply this criterion to the previous minimax problem, we consider a function

$$\varphi(x, a) = | F(z^1(x, a), z^2(x, a), \ldots, z^n(x, a);\ x;\ a) |$$

Components of the vector $Y^{(\nu)}$ are given by

$$y_k^{(\nu)} = \Big[\Big(\sum_{j=1}^{n} F_{z^j} p_k^{\,j} + F_{a_k}\Big) \operatorname{sign} F\Big]_{x=x^{(\nu)},\ a=a^*} \tag{1.16}$$

Using Formula (1, 11) we eliminate $p_k^{\,j}$ and insert the result into (1, 14) to obtain the basic system

$$\sum_\nu m_\nu \Big\{ \sum_{j=1}^{n} F_{z^j} \sum_{i=1}^{n} \Big[ (\lambda_i^{j(\nu)} p_k^{\,i})_{x=0} + \int_0^{x^{(\nu)}} \lambda_i^{j(\nu)} f_{a_k}^{\,i}\, dx \Big] + F_{a_k} \Big\}_{x=x^{(\nu)}} \operatorname{sign} F(x^{(\nu)}) = 0$$

$$(a = a^\circ,\ k = 1, \ldots, r) \tag{1.17}$$

Formula (1, 6) yields an expression for $(p_k^{\,i})_{x=0}$ in terms of parameters; points $x^{(\nu)}$ are found from equations expressing the fact that the total derivative of $\varphi(x, a)$ with respect to $x$, is equal to zero.

The requirement that (1, 17) has positive solutions replaces now the condition (1, 9) and the corresponding second order condition in the Meyer problem.

**2.** In a number of cases, well known results of the formal theory of functions make it possible to bypass the direct investigation of the system (1, 17). As an example, let us consider Eq.

$$dz / dx = z + a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \tag{2.1}$$

We must choose the coefficients $a_i$ ($i = 0, 1, \ldots, n$) and the initial value of $z(0) = a_{n+1}$ so, that the corresponding solution $z(x)$ would exhibit, over the interval $[0, l]$, a minimum deviation from the given continuous function $f(x)$

$$\max | z(x) - f(x) | = \min$$

Multiplier $\lambda$ is not required here and Eq. (2, 1) is integrable. Solution can be expressed in form of linear combination of functions

$$e^x,\ 1,\ x,\ x^2,\ \ldots\ x^n \tag{2.2}$$

with coefficients in form of linear combinations of parameters $a_i$ ($i = 0, \ldots, n + 1$).

As we know, the set (2, 2) forms a Chebyshev system on any finite interval ([3], p. 13), therefore we can find the coefficients giving minimum deviation, using a rule following from the fundamental Chebyshev theorem ([3], pp. 16 to 20). Direct investigation of (1, 17) would, in this case, lead to establishing a fundamental theorem for (2, 2) in a manner similar to that employed in [1 and 2] for the power system.

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